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Hulek, K. (D-HANN-IM); **Spandaw, J.** (D-HANN-IM); **van Geemen, B.** (I-PAVI);
van Straten, D. (D-MNZ)
The modularity of the Barth-Nieto quintic and its relatives. (English summary)
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The Barth-Nieto quintic is defined by the equations $N=\{\sum_{i=0}^5 x_i=\sum_{i=0}^5 \frac{1}{x_i}=0\}$ in $\mathbf{P}^5$. W. P. Barth and I. Nieto [J. Algebraic Geom. **3** (1994), no. 2, 173–222; MR1257320 (95e:14033)] showed that a double cover of N and N have Calabi-Yau models Z and Y, respectively, where Y is defined over $\mathbf{Z}$.

In this paper, the authors show that both of these Calabi-Yau threefolds are rigid, and then establish their modularity by determining the L-series explicitly. To this end, replace Z by another model $\tilde{Y}$ which is defined over $\mathbf{Z}$ and which has $H^3(\tilde{Y})\simeq H^3(Z)$. Theorem 1: $L(H^3_{\text{et}}(Y),s)=L(H^3_{\text{et}}(\tilde{Y}),s)=L(f,s)$ where f is a unique normalized newform of weight 4 on the congruence subgroup $\Gamma_0(6)$, $f(q)=[\eta(q)\eta(q^2)\eta(q^3)\eta(q^6)]^2$. The equalities hold up to Euler factors corresponding to primes 2 and 3.

H. A. Verrill [J. Number Theory **81** (2000), no. 2, 310–334; MR1752257 (2002j:14026)] constructed the rigid Calabi-Yau threefold V_{A_3} associated to the root lattice A_3 and showed that its L-series coincides with the L-series of the weight 4 cusp form f on $\Gamma_0(6)$. Her method was to make use of the Serre-Faltings criterion.

The Tate conjecture implies that there should be an algebraic correspondence defined over $\mathbf{Q}$ between Y and V_{A_3} which gives rise to the equivalent 2-dimensional Galois representations, and hence the same L-series.

In fact, N. Yui [in *Arithmetic algebraic geometry (Park City, UT, 1999)*, 507–569, Amer. Math. Soc., Providence, RI, 2001 MR1860046 (2003d:11093)] raised the question whether there is connection between the Barth-Nieto quintic and the Verrill rigid Calabi-Yau threefolds.

M.-H. Saitō and N. Yui [J. Math. Kyoto Univ. **41** (2001), no. 2, 403–419; MR1852991 (2002k:11102)] established the modularity of V_3 by constructing explicitly a birational map defined over $\mathbf{Q}$ between V_{A_3} and the self-fiber product, $W:=S_1(6)\times_{X_1(6)}S_1(6)$, of the universal elliptic curve $S_1(6)$ over the modular curve $X_1(6)$. From the construction, the L-series of W is given by the cusp form f of weight 4 on $\Gamma_1(6)$ (which coincides with its projectivization $\Gamma_0(6)$).

A smooth projective variety X defined over $\mathbf{Q}$ is called a relative of the Barth-Nieto quintic if the l-adic Galois representation associated to the newform f in Theorem 1 occurs in $H^3_{\text{et}}(X)$. The rigid Calabi-Yau threefolds W and V_{A_3} are relatives of Y. The main result of this paper is Theorem 2: There exist birational maps defined over $\mathbf{Q}$ between V_{A_3} and Y, and between W and Y.

Combining these birational maps, the authors obtain a birational map between W and V_{A_3} over $\mathbf{Q}$, which is different from the one constructed by Saito and Yui.

Reviewed by *Noriko Yui*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.